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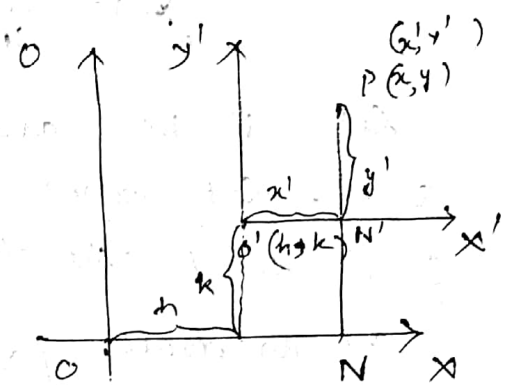
Transformation of coordinates:

Change of origin without changing the direction of the axes:



Let us take a new pair of coordinate axes O'X' and O'Y' parallel to the old axes OX and OY, O' being the new origin whose co-ordinates w.r.t. the old axes are (h, k), say.

Let P be any point in the plane of the axes. Let (x, y) and (x', y') be the co-ordinates of P w.r.t. the old and new axes respectively.

Let us draw PN ⊥ OX meeting O'X' at N'.

Then $x = ON$
$= x' + h$

and $y = PN = PN' + N'N = y' + k$

$\therefore \quad x = x' + h, \quad y = y' + k.$

which are the transformation formulae.

Transformation from one pair of rectangular axes to another with the same origin; (Rotation of axes).

Suppose the axes OX and OY are rotated through an angle θ, and OX' and OY' be the new positions of the axes.

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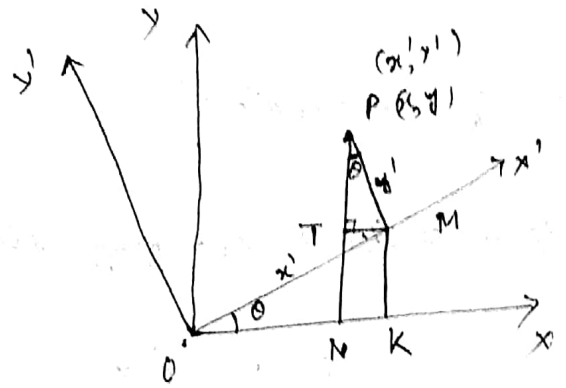

Let P be any point in the plane of axes. Let (x, y) and (x', y') be the co-ordinates of P w.r.t. the old and new axes respectively.

Let us draw $PN \perp OX$, $PM \perp OX'$, $MK \perp OX$, and $MT \perp PN$.

From figure, we get,

$$x = ON = OK - NK$$
$$= OK - MT$$
$$= OM\cos\theta - PM\sin\theta \qquad \left[\because \frac{OK}{OM} = \cos\theta,\ \frac{MT}{PM} = \sin\theta\right]$$
$$\Rightarrow x = x'\cos\theta - y'\sin\theta$$

~~Again~~, And,

$$y = PN = PT + TN$$
$$= PT + MK$$
$$= PM\cos\theta + OM\sin\theta \qquad \left[\frac{PT}{PM} = \cos\theta,\ \frac{MK}{OM} = \sin\theta\right]$$
$$= y'\cos\theta + x'\sin\theta$$

Thus

$$\left.\begin{aligned} x &= x'\cos\theta - y'\sin\theta \\ y &= x'\sin\theta + y'\cos\theta \end{aligned}\right\}$$

which are the transformation formulae.

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Invariants: If by rotation of the rectangular axes about the origin, the expression $ax^2+2hxy+by^2$ changes to $a'x'^2+2h'x'y'+b'y'^2$, then

$$a+b = a'+b'$$
$$\text{and } ab-h^2 = a'b'-h'^2$$

* If ax^2+bx+c is a perfect square, then $ax^2+bx+c=0$ will have equal roots
for this $b^2-4ac=0$
$\therefore$ Here $(a+\lambda)x^2+2hxy+(b+\lambda)y^2$,
$(2hy)^2-4(a+\lambda)(b+\lambda)y^2=0 \Rightarrow (a+\lambda)(b+\lambda)-h^2=0$

Pf.- Here, ~~Let~~ (x,y) and (x',y') are the co-ordinates of any point P, say, w.r.t. the old and new axes respectively. Then the expression x^2+y^2 changes to $x'^2+y'^2$, because each represents OP^2.

Then the expression $ax^2+2hxy+by^2+\lambda(x^2+y^2)$ changes to $a'x'^2+2h'x'y'+b'y'^2+\lambda(x'^2+y'^2)$, where λ is any constant.

$\therefore$ If for some λ, the expression $ax^2+2hxy+by^2+\lambda(x^2+y^2)$ becomes a perfect square, then for the same value of λ, the other expression $a'x'^2+2h'x'y'+b'y'^2+\lambda(x'^2+y'^2)$ will also become a perfect square.

Now, $ax^2+2hxy+by^2+\lambda(x^2+y^2)$
$= (a+\lambda)x^2+2hxy+(b+\lambda)y^2$
is a perfect square if

$$(a+\lambda).(b+\lambda)-h^2=0.$$

* $(lx+my)^2$
$= l^2x^2+2lmxy+m^2y^2$
$\therefore l^2m^2=(lm)^2$
$\Rightarrow l^2m^2-(lm)^2=0$

i.e. if $\lambda^2+\lambda(a+b)+(ab-h^2)=0$ —— (1)

Similarly, the other expression i.e.
$(a'+\lambda)x'^2+2h'x'y'+(b'+\lambda)y'^2$ is a perfect square if $\lambda^2+\lambda(a'+b')+(a'b'-h'^2)=0$ —— (2)

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Now, the two quadratic equations in λ, namely (1) and (2) must give the same values of λ. Hence the co-efficients must be proportional.

$$\therefore \quad \frac{1}{1} = \frac{a+b}{a'+b'} = \frac{ab-h^2}{a'b'-h'^2}$$

$$\Rightarrow \quad \frac{a+b}{a'+b'} = 1 \quad \text{and} \quad \frac{ab-h^2}{a'b'-h'^2} = 1$$

$$\Rightarrow \quad a+b = a'+b' \qquad \Rightarrow ab-h^2 = a'b'-h'^2.$$

These two quantities $a+b$ and $ab-h^2$ for the expression $ax^2+2hxy+by^2$, are called invariants.

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Removal of the xy-term:

To find the angle through which the axes are to be rotated in order to remove the term containing xy, from the expression $ax^2+2hxy+by^2$.

If the axes are rotated through an angle θ, then the expression $ax^2+2hxy+by^2$ changes to

$$a(x'\cos\theta - y'\sin\theta)^2 + 2h(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta) + b(x'\sin\theta + y'\cos\theta)^2$$

If in the new expression the term containing $x'y'$ is vanishing, then co-efficient of $x'y'$ will be zero.

$$\therefore \quad -a.2\cos\theta\sin\theta + 2h.(\cos^2\theta - \sin^2\theta) + b.2\sin\theta\cos\theta = 0$$

$$\Rightarrow \quad -a\sin 2\theta + 2h.\cos 2\theta + b.\sin 2\theta = 0$$

$$\Rightarrow \quad (b-a)\sin 2\theta = -2h\cos 2\theta.$$

$$\Rightarrow \quad \frac{\sin 2\theta}{\cos 2\theta} = \frac{-2h}{b-a}$$

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$\Rightarrow \tan 2\theta = \frac{2h}{a-b}$

$\Rightarrow 2\theta = \tan^{-1}\frac{2h}{a-b}$

$\Rightarrow \boxed{\theta = \frac{1}{2}\tan^{-1}\frac{2h}{a-b}}$

Ex. If by rotation of the rectangular axes, the equation $17x^2+18xy-7y^2=1$ be reduced to the form $ax^2+by^2=1$, find a and b; find also the angle through which the axes are rotated.

Soln. By invariants, we get

$a+b = 17+(-7)$ $\quad [a'+b' = a+b$

$= 10$ ——— (1)

And $ab-h^2 = 17.(-7)-9^2$ $\quad [a'b'-h'^2 = ab-h^2]$

$\Rightarrow ab = -119-81$

$= -200$

$\therefore a-b = \sqrt{(a+b)^2-4ab}$

$= \sqrt{10^2-4\times(-200)}$

$= \sqrt{100+800}$

$= \sqrt{900}$

$\Rightarrow a-b = 30$ ——— (2)

$\therefore$ (1)+(2) $\Rightarrow 2a = 40$

$\Rightarrow a = 20.$

& (1)−(2) $\Rightarrow 2b = -20$

$\Rightarrow b = -10$

Since xy term is removed from the given

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expression, therefore if the angle of rotation

∴ ~~The angle~~

is θ, then

$$\theta = \frac{1}{2}\tan^{-1}\frac{2h}{a-b}$$

$17x^2+18xy-7y^2=1$

$$= \frac{1}{2}\tan^{-1}\frac{2\times 9}{17-(-7)}$$

$$= \frac{1}{2}\tan^{-1}\frac{\cancel{18}\,3}{\cancel{24}\,4}$$

$$= \frac{1}{2}\tan^{-1}\frac{3}{4}.$$

Ex. Find the angle through which a set of rectangular axes must be turned without the change of origin so that the expression $7x^2+4xy+3y^2$ will be transformed into the form $a'x^2+b'y^2$.

Soln. The xy term vanishes from given expression.

∴ The angle of rotation is given by

$$\theta = \frac{1}{2}\tan^{-1}\frac{2h}{a-b}$$

$$= \frac{1}{2}\tan^{-1}\frac{2\cdot 2}{7-3}$$

$$= \frac{1}{2}\tan^{-1}\frac{4}{4}$$

$$= \frac{1}{2}\tan^{-1} 1$$

$$= \frac{1}{2}\cdot\frac{\pi}{4}$$

$$= \frac{\pi}{8}$$

$$= \frac{180^\circ}{8}$$

$$= 22\tfrac{1}{2}^\circ$$

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Ex. Remove the xy term from the equn $3x^2+2xy+3y^2-2=0$.

Soln. The given equn is

$$3x^2+2xy+3y^2-2=0 \quad \text{--- (1)}$$

Let the transformed equn be

$$ax^2+by^2-2=0. \quad \text{--- (2)}$$

$\therefore$ By invariants,

$$a+b = 3+3$$
$$= 6 \quad \text{--- (3)}$$

and, $ab-0^2 = 3.3-1^2$

$\Rightarrow ab = 8$

$\therefore a-b = \sqrt{(a+b)^2-4ab}$.

$= \sqrt{6^2-4\times 8}$

$= \sqrt{4}$

$\Rightarrow a-b = 2 \quad \text{--- (4)}$

$\therefore$ (3) + (4) $\Rightarrow 2a = 8$

$\Rightarrow a = 4$.

and (3) − (4) $\Rightarrow 2b = 4$

$\Rightarrow b = 2$.

$\therefore$ (2) becomes,

$4x^2+2y^2-2=0$

$\Rightarrow 2x^2+y^2 = 1$ //

Note: The angle of rotation is given by

$$\theta = \frac{1}{2}\tan^{-1}\frac{2h}{a-b} = \frac{1}{2}\tan^{-1}\frac{2.1}{3-3} = \frac{1}{2}\tan^{-1}\frac{2}{0} = \frac{1}{2}\tan^{-1}\infty$$
$$= \frac{1}{2}\times 90^\circ$$
$$= 45^\circ \text{ //}$$

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